

# COXETER GROUPS AND AUTOMORPHISMS

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**ABSTRACT.** Let  $(W, S)$  be a Coxeter system and  $\Gamma$  be a group of automorphisms of  $W$  such that  $\gamma(S) = S$  for all  $\gamma \in \Gamma$ . Then it is known that the group of fixed points  $W^\Gamma$  is again a Coxeter group with a canonically defined set of generators. The usual proofs of this fact rely on the reflection representation of  $W$ . Here, we give a proof which only uses the combinatorics of reduced expressions in  $W$ . As a by-product, this shows that the length function on  $W$  restricts to a weight function on  $W^\Gamma$ .

Let  $(W, S)$  be a Coxeter system where  $S$  is finite. Let  $l: W \rightarrow \mathbb{N}_0$  be the corresponding length function. Let  $\Gamma$  be a group of automorphisms of  $W$  such that  $\gamma(S) = S$  for all  $\gamma \in \Gamma$ . Then we have  $l(\gamma(w)) = l(w)$  for all  $w \in W$ . Let

$$W^\Gamma := \{w \in W \mid \gamma(w) = w \text{ for all } \gamma \in \Gamma\}$$

be the group of fixed points. For any subset  $I \subseteq S$  let  $W_I \subseteq W$  be the corresponding parabolic subgroup. If  $W_I$  is finite, let  $w_I \in W_I$  be the longest element. Let  $\bar{S}$  be the set of  $\Gamma$ -orbits  $I$  on  $S$  such that  $W_I$  is finite; note that  $w_I \in W^\Gamma$  for  $I \in \bar{S}$ . The purpose of this note is to provide a short proof of the following (known) result.

**Theorem 1.** *The pair  $(W^\Gamma, \{w_I \mid I \in \bar{S}\})$  is a Coxeter system. Let  $l_\Gamma: W^\Gamma \rightarrow \mathbb{N}_0$  be the corresponding length function. Then, for any  $w, w' \in W^\Gamma$ , we have  $l(ww') = l(w) + l(w')$  if and only if  $l_\Gamma(ww') = l_\Gamma(w) + l_\Gamma(w')$ .*

If  $W$  is finite, this is due to Steinberg [6, §11]; for general  $(W, S)$ , see Hée [3] or Lusztig [5, Appendix]. The proofs in [*loc. cit.*] rely on properties of the reflection representation of  $W$ . The proof that we shall give here is based on notes of a course on Coxeter groups given by the second-named author at EPFL in 2004. It proceeds somewhat more directly by using only the combinatorics of reduced expressions of elements in  $W$ .

A key role is played by dihedral groups and distinguished coset representatives with respect to a parabolic subgroup of  $W$  (see, for example, [2, §2.1]). Also recall that a parabolic subgroup  $W_I$  is finite if and only if there exists an element  $u \in W_I$  such that  $l(su) < l(u)$  for all  $s \in I$ , in which case we have  $u = w_I$ ; note also that  $w_I^2 = 1$ . (For these facts see, for example, [2, §1.5].)

**Lemma 2** (Cf. [3, 3.4], [5, A.1(a)]). *Let  $w \in W^\Gamma$ . Then we can write  $w = w_{J_1} \cdots w_{J_r}$  where  $J_i \in \bar{S}$  and  $l(w) = l(w_{J_1}) + \dots + l(w_{J_r})$ . Furthermore, if  $s \in S$  is such that  $l(sw) < l(w)$ , then we can choose  $J_1$  such that  $s \in J_1$ . In particular,  $W^\Gamma = \langle w_I \mid I \in \bar{S} \rangle$ .*

*Proof.* Induction on  $l(w)$ . If  $l(w) = 0$ , then  $w = 1$  and there is nothing to prove. Now let  $l(w) > 0$  and  $s \in S$  be such that  $l(sw) < l(w)$ . Let  $J_1$  be the  $\Gamma$ -orbit of  $s$ ; since  $l(sw) < l(w)$ , we have  $l(\gamma(s)w) = l(\gamma(sw)) < l(w)$  for all  $\gamma \in \Gamma$  and, hence,  $l(tw) < l(w)$  for all  $t \in J_1$ . Let now  $X_{J_1} = \{x \in W \mid l(tx) > l(x) \text{ for all } t \in J_1\}$  be the set of distinguished right coset representatives of  $W_{J_1}$  in  $W$ . We can write  $w = ux$  where  $u \in W_{J_1}$ ,  $x \in X_{J_1}$  and  $l(w) = l(u) + l(x)$ . Since  $tu \in W_{J_1}$ , we have  $l(tux) = l(tu) + l(x)$  for all  $t \in J_1$ . So we conclude that  $l(tu) < l(u)$  for all  $t \in J_1$ . Hence,  $W_{J_1}$  must be finite and  $u = w_{J_1} \in W^\Gamma$ . But then we also have  $x \in W^\Gamma$  and we can continue with  $x$  by induction.  $\square$

In what follows, to simplify notation, we shall write  $w = x \bullet y$  if  $w, x, y \in W$  are such that  $w = xy$  and  $l(w) = l(x) + l(y)$ . Thus, in the setting of Lemma 2, we can write  $w = w_{J_1} \bullet \dots \bullet w_{J_r}$ .

*Remark 3.* Let  $I, J \in \bar{S}$  and assume that  $I \neq J$ . Let  $K := I \cup J$ . Applying Lemma 2 to  $W_K$  shows that  $W_K^\Gamma = \langle w_I, w_J \rangle$  is a dihedral group. Suppose that there exists some  $u \in W_K$  such that  $l(su) < l(u)$  for all  $s \in K$ . Then  $W_K$  is finite and  $u = w_K \in W_K^\Gamma$ . Being a dihedral group, the order of  $W_K^\Gamma$  is  $2m$  for some  $m \in \mathbb{N}$ . The elements of  $W_K^\Gamma$  are products of the form  $w_I w_J w_I \dots$  or  $w_J w_I w_J \dots$ , with at most  $m$  factors; furthermore, two such products (one starting with  $w_I$  and one starting with  $w_J$ ) are equal if and only if there are exactly  $m$  factors in each of them. This also shows that  $l(y) \leq \frac{m}{2}(l(w_I) + l(w_J))$  for all  $y \in W_K^\Gamma$ . We now claim that

$$w_K = \underbrace{w_I \bullet w_J \bullet w_I \bullet \dots}_{m \text{ terms}} = \underbrace{w_J \bullet w_I \bullet w_J \bullet \dots}_{m \text{ terms}} \quad \text{where} \quad l(w_K) = \frac{m}{2}(l(w_I) + l(w_J)).$$

Indeed, by Lemma 2, we can write  $w_K = w_I \bullet w_J \bullet w_I \bullet \dots$  with, say  $p \geq 1$ , terms, and also  $w_K = w_J \bullet w_I \bullet w_J \bullet \dots$  with, say  $q \geq 1$ , terms. Since  $l(w_K) \leq \frac{m}{2}(l(w_I) + l(w_J))$ , we must have  $p \leq m$  and  $q \leq m$ . But then the two products  $w_K = w_I \bullet w_J \bullet w_I \bullet \dots = w_J \bullet w_I \bullet w_J \bullet \dots$  can only be equal if there are exactly  $m$  factors in both sides. Thus, we have  $p = q = m$ , as required.

**Lemma 4.** *Let  $w \in W^\Gamma$  and assume that we have two expressions*

$$w = w_{J_1} \bullet \dots \bullet w_{J_r} = w_{I_1} \bullet \dots \bullet w_{I_p}$$

*where  $J_i, I_i \in \bar{S}$ . Then  $r = p$ .*

*Proof.* Induction on  $l(w)$ . If  $l(w) = 0$ , then  $w = 1$  and there is nothing to prove. Now assume that  $l(w) > 0$ ; then  $r \geq 1$  and  $p \geq 1$ . If  $I_1 = J_1$ , then  $w' := w_{I_1} w = w_{J_2} \bullet \dots \bullet w_{J_r} = w_{I_2} \bullet \dots \bullet w_{I_p}$ . So  $r - 1 = p - 1$  by induction. Now assume that  $I_1 \neq J_1$ . Let  $K := I_1 \cup J_1$  and  $X_K = \{x \in W \mid l(sx) > l(x) \text{ for all } s \in K\}$  be the set of distinguished coset representatives of  $W_K$  in  $W$ . We can write  $w = u \bullet x$  where  $u \in W_K$  and  $x \in X_K$ . We have  $l(sw) < l(w)$  for all  $s \in K$  and so  $l(su) < l(u)$  for all  $s \in K$ . Hence, by Remark 3,  $W_K$  must be finite and  $u = w_K \in W_K^\Gamma$ . Then we also have  $x \in W^\Gamma$  and so, by Lemma 2, we can write  $x = w_{L_1} \bullet \dots \bullet w_{L_q}$  where  $L_i \in \bar{S}$ . Now consider the identities:

$$w_{J_1} \bullet \dots \bullet w_{J_r} = w = w_K \bullet x = \underbrace{(w_{J_1} \bullet w_{I_1} \bullet w_{J_1} \bullet \dots)}_{m \text{ terms}} \bullet w_{L_1} \bullet \dots \bullet w_{L_q}.$$

Cancelling  $w_{J_1}$  on the left on both sides, we deduce that

$$w_{J_2} \bullet \dots \bullet w_{J_r} = \underbrace{(w_{I_1} \bullet w_{J_1} \bullet \dots)}_{m-1 \text{ terms}} \bullet w_{L_1} \bullet \dots \bullet w_{L_q}.$$

By induction, we conclude that  $r - 1 = (m - 1) + q$ . Applying the same argument to

$$w_{I_1} \bullet \dots \bullet w_{I_p} = w = w_K \bullet x = \underbrace{(w_{I_1} \bullet w_{J_1} \bullet w_{I_1} \bullet \dots)}_{m \text{ terms}} \bullet w_{L_1} \bullet \dots \bullet w_{L_q}.$$

also yields  $p - 1 = (m - 1) + q$ . Consequently, we obtain  $r = p$ , as desired.  $\square$

The above proof is inspired by the proof of the Matsumoto–Tits Lemma in [2, 1.2.2], which in turn follows Tits [7, Cor. II.1.12].

Now let  $\lambda: W^\Gamma \rightarrow \mathbb{N}_0$  denote the length function with respect to the generators  $\{w_I \mid I \in \bar{S}\}$ . Thus, the properties in [1, Chap. IV, n° 1.1] do hold for  $\lambda$  but note, for example, that it is not yet clear that  $\lambda(w_I w) \neq \lambda(w)$  for  $w \in W^\Gamma$  and  $I \in \bar{S}$ .

**Lemma 5.** *Let  $w \in W^\Gamma$  and  $w = w_{J_1} \cdots w_{J_p}$  where  $J_i \in \bar{S}$ . If  $p = \lambda(w)$ , then  $w = w_{J_1} \bullet \dots \bullet w_{J_p}$ .*

*Proof.* Induction on  $p$ . If  $p = 0$  or  $1$ , then the assertion is clear. Now assume that  $p \geq 2$  and set  $w' := w_{J_2} \cdots w_{J_p} \in W^\Gamma$ . Then  $\lambda(w') = p - 1$  and so, by induction,  $w' = w_{J_2} \bullet \dots \bullet w_{J_p}$ . Now we distinguish two cases. If  $l(sw') > l(w')$  for all  $s \in J_1$ , then  $w' \in X_{J_1}$  and so  $l(w_{J_1} w') = l(w_{J_1}) + l(w')$ . Thus,  $w = w_{J_1} \bullet w'$  and the desired assertion is proved. On the other hand, if  $l(sw') < l(w')$  for some  $s \in J_1$ , then we can also find an expression  $w' = w_{L_1} \bullet \dots \bullet w_{L_q}$  where  $L_i \in \bar{S}$  and  $L_1 = J_1$ ; see Lemma 2. By Lemma 4, we have  $p - 1 = q$ . Now  $w = w_{J_1} w' = w_{L_2} \cdots w_{L_{p-1}}$  and so  $\lambda(w) < p$ , a contradiction. Hence, this case does not occur.  $\square$

**Corollary 6.** *Let  $w, w' \in W^\Gamma$ . Then  $l(w w') = l(w) + l(w')$  if and only if  $\lambda(w w') = \lambda(w) + \lambda(w')$ . In particular, the restriction of  $l$  to  $W^\Gamma$  is a weight function in the sense of Lusztig [5].*

*Proof.* Let  $r = \lambda(w)$  and  $p = \lambda(w')$ . By Lemma 5, we have  $w = w_{J_1} \bullet \dots \bullet w_{J_r}$  and  $w' = w_{I_1} \bullet \dots \bullet w_{I_p}$  where  $J_i, I_i \in \bar{S}$ .

Assume first that  $l(w w') = l(w) + l(w')$ . Then  $w w' = w_{J_1} \bullet \dots \bullet w_{J_r} \bullet w_{I_1} \bullet \dots \bullet w_{I_p}$ . Let  $q := \lambda(w w') \leq r + p$ . Again, by Lemma 5, we have  $w w' = w_{L_1} \bullet \dots \bullet w_{L_q}$  where  $L_i \in \bar{S}$ . Now Lemma 4 implies that  $q = r + p$ , as desired. Conversely, assume that  $\lambda(w w') = \lambda(w) + \lambda(w')$ . Since we have  $w w' = w_{J_1} \cdots w_{J_r} w_{I_1} \cdots w_{I_p}$ , Lemma 5 shows once more that  $w w' = w_{J_1} \bullet \dots \bullet w_{J_r} \bullet w_{I_1} \bullet \dots \bullet w_{I_p}$ . Thus, we have  $l(w w') = l(w) + l(w')$ .  $\square$

Since  $(W, S)$  is a Coxeter system, the "Exchange Condition" holds. Recall that this means the following. Let  $w \in W$  and  $s \in S$ . Let  $p = l(w)$  and  $w = s_1 \cdots s_p$  where  $s_i \in S$ . If  $l(sw) \leq l(w)$ , then there exists some  $i \in \{1, \dots, p\}$  such that  $sw = s_1 \cdots s_{i-1} s_{i+1} \cdots s_p$ . We can now show that the pair  $(W^\Gamma, \{w_I \mid I \in \bar{S}\})$  also satisfies this "Exchange Condition" and, hence,  $(W^\Gamma, \{w_I \mid I \in \bar{S}\})$  is a Coxeter system; see Bourbaki [1, Chap. IV, n° 1.6]. In combination with Corollary 6, this will complete the proof of Theorem 1.

**Proposition 7.** *Let  $w \in W^\Gamma$  and  $I \in \bar{S}$ . Let  $p = \lambda(w)$  and  $w = w_{J_1} \cdots w_{J_p}$  where  $J_i \in \bar{S}$ . If  $\lambda(w_I w) \leq \lambda(w)$ , then there exists some  $i \in \{1, \dots, p\}$  such that  $w_I w = w_{J_1} \cdots w_{J_{i-1}} w_{J_{i+1}} \cdots w_{J_p}$ .*

*Proof.* If we had  $l(sw) > l(w)$  for all  $s \in I$ , then  $w \in X_I$  and so  $l(w_I w) = l(w_I) + l(w)$ . Hence, Corollary 6 would imply that  $\lambda(w_I w) = \lambda(w_I) + \lambda(w) > \lambda(w)$ , contrary to our assumption. Thus, there exists some  $s \in I$  such that  $l(sw) \leq l(w)$ . Further note that, by Lemma 5, we have  $w = w_{J_1} \bullet \dots \bullet w_{J_p}$ . Taking reduced expressions for all  $w_{J_i}$ , we obtain a reduced expression for  $w$ . Since the "Exchange Condition" holds for  $(W, S)$ , there exists an index  $i \in \{1, \dots, p\}$  such that

$$sw = w_{J_1} \cdots w_{J_{i-1}} x w_{J_{i+1}} \cdots w_{J_p}$$

where  $x \in W_{J_i}$  is obtained by dropping one factor in a reduced expression for  $w_{J_i}$ . Consequently, we have

$$z^{-1}sz = xw_{J_i} \quad \text{where} \quad z := w_{J_1} \cdots w_{J_{i-1}}.$$

Since  $z \in W^\Gamma$ , we obtain  $z^{-1}\gamma(s)z = \gamma(xw_{J_i}) \in W_{J_i}$  for all  $\gamma \in \Gamma$  and so  $z^{-1}w_I z \in W_{J_i}$ . Since also  $z^{-1}w_I z \in W^\Gamma$  and  $W_{J_i}^\Gamma = \{1, w_{J_i}\}$  (see Lemma 2), we conclude that  $z^{-1}w_I z = w_{J_i}$ . This yields  $w_I w_{J_1} \cdots w_{J_{i-1}} = w_{J_1} \cdots w_{J_{i-1}} w_{J_i}$  and so  $w_I w = w_{J_1} \cdots w_{J_{i-1}} w_{J_{i+1}} \cdots w_{J_p}$ , as required.  $\square$

*Remark 8.* Using similar arguments, one can extend Theorem 1 to the following "relative" setting (see [4, §5] and [5, Chap. 25]). We fix a subset  $I_0 \subseteq S$  such that  $W_{I_0}$  is finite and  $\gamma(I_0) = I_0$  for all  $\gamma \in \Gamma$ ; furthermore, we assume that

$$w_{I_0 \cup J} \in N_W(W_{I_0}) \quad \text{for all } J \in \mathcal{S},$$

where  $\mathcal{S}$  denotes the set of all  $\Gamma$ -orbits on  $S \setminus I_0$  such that  $W_{I_0 \cup J}$  is finite. Let

$$\mathcal{W} := \{w \in X_{I_0} \mid wW_{I_0} = W_{I_0}w\}.$$

Then  $\mathcal{W}$  is a subgroup of  $W$  such that  $\gamma(\mathcal{W}) = \mathcal{W}$  for all  $\gamma \in \Gamma$ . Hence, we can consider the group of fixed points  $\mathcal{W}^\Gamma$ . We set

$$s_J := w_{I_0 \cup J} w_{I_0} = w_{I_0} w_{I_0 \cup J} \quad \text{for each } J \in \mathcal{S}.$$

Then one can show that  $s_J \in \mathcal{W}^\Gamma$  and  $s_J^2 = 1$ ; furthermore,  $(\mathcal{W}^\Gamma, \{s_J \mid J \in \mathcal{S}\})$  is a Coxeter system. (Theorem 1 is the special case where  $I_0 = \emptyset$ .) We omit further details.

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